Multiplicity, Invariants and Tensor Product Decompositions of Tame Representations of U(∞)

R. Michael Howe
Department of Mathematics
University of Wisconsin-Eau Claire
Eau Claire, Wisconsin 54702
email: hower@uwec.edu

Tuong Ton-That
Department of Mathematics
University of Iowa
Iowa City, Iowa 52242
email: tonthat@math.uiowa.edu

February 7, 2008

Abstract

The structure of r-fold tensor products of irreducible tame representations of $U(\infty) = \varinjlim U(n)$ are described, versions of contragredient representations and invariants are realized, and methods of calculating multiplicities, Clebsch-Gordan and Racah coefficients are given using invariant theory on Bargmann-Segal-Fock spaces.

PACS codes: 02.20.Tw, 02.20.QS, 03.65.Fd

I Introduction

Let G_k and $G_k^{\mathbb{C}}$ denote the unitary group and the general linear group, respectively. Then the *inductive limits*

$$G_\infty = \varinjlim\ G_k = \bigcup_{k=1}^{\infty} G_k$$

and

$$G_\infty^{\mathbb{C}} = \varinjlim\ G_k^{\mathbb{C}} = \bigcup_{k=1}^{\infty} G_k^{\mathbb{C}}$$

may be defined as follows:

$G_\infty^{\mathbb{C}} = \{g = (g_{ij})_{i,j\in\mathbb{N}} \mid g \text{ is invertible and all but a finite number of } g_{ij} - \delta_{ij} = 0\}$

and

$$G_\infty = \{u \in G_\infty^{\mathbb{C}} \mid u^* = u^{-1}\}.$$

Representation theory of G_∞ and $G_\infty^{\mathbb{C}}$ was first studied by I. Segal in [1], then by A. Kirillov in [2], followed by S. Stratila and D. Voiculescu in [3], D. Pickerell in [4], G. Ol'shanskii in [5] [6] [7], I. Gelfand and M. Graev in [8], and V. Kac in [9]. This list is certainly not exhaustive, and the most complete list of references can be found in the comprehensive and important work of Ol'shanskii.

Following Ol'shanskii we call a unitary representation of G_∞ *tame* if it is continuous in the group topology in which the descending chain of subgroups of the type $\{\left(\begin{smallmatrix} 1_k & 0 \\ 0 & * \end{smallmatrix}\right)\}$, $k = 1, 2, 3, \ldots$ constitutes a fundamental system of neighborhoods of the identity 1_∞. Assume that for each k a unitary representation (R_k, H_k) of G_k is given and an isometric embedding (of Hilbert spaces) $i_{k+1}^k : H_k \longrightarrow H_{k+1}$ commuting with the action of G_k (i.e., $i_{k+1}^k \circ R_k(u) = R_{k+1}(u) \circ i_{k+1}^k$) is given. If H_∞ denotes the Hilbert space completion of $\bigcup_{k=1}^{\infty} H_k$, then there exists uniquely a unitary representation

R_∞ of G_∞ on H_∞ defined by

$$R_\infty(u)f = R_k(u)f \quad \text{if} \quad u \in G_k \text{ and } f \in H_k \ .$$

The representation (R_∞, H_∞) is called the *inductive limit* of the sequence (R_k, H_k), and we have the following Theorem (see [5] for a proof).

Theorem I.1. *If the representations (R_k, H_k) are all irreducible then the inductive limit (R_∞, H_∞) is also irreducible.*

Let

$$\lambda_{G_k} = (m_1, \dots, m_k), \quad m_1 \geq m_2 \geq \dots \geq m_k \geq 0, \quad m_i \in \mathbb{N} \cup \{0\}, \quad i = 1 \dots k.$$

then Ol'shanskii proved the following

Theorem I.2. *All unitary irreducible tame representations of G_∞ are the inductive limits of the sequences of the form $\{\rho_\lambda, V^{\lambda_{G_k}}\}$, where in each $(\lambda) = (m_1, m_2, \dots)$ the m_i are equal to 0 for sufficiently large i.*

It follows from 'Weyl's unitarian trick' that all irreducible tame representations of G_∞ are inductive limits of sequences of the form $\{\rho_\lambda, V^{\lambda_{G_k}}\}$. Following Ol'shanskii a representation of G_∞ is called *holomorphic* if it is a direct sum (of any number) of irreducible tame representations.

In this paper we consider the problem of decomposing an r-fold tensor product of unitary irreducible tame representations of G_∞. Such a problem was investigated in [2] and [8] for the simplest type of tame irreducible representations, namely the fundamental (or principal) ones. In light of the

recent interest in Physics in the representation theory of U_∞ it is natural to consider such an important problem in this theory.

The general problem can be stated as follows: Given r tame irreducible $(G_\infty, V^{(\lambda_i)^\infty})$ modules, choose a basis $|\lambda_i, \xi_i\rangle$ for each i (such a basis always exists, for example, the generalized Gelfand-Žetlin basis given in [8], but we do not limit ourselves only to this basis). Form the r-fold tensor product $(\lambda_1)^\infty \otimes \cdots \otimes (\lambda_r)^\infty$ and calculate the number of times the irreducible representation $(\lambda)^\infty$ occurs in the tensor product. The first method to compute this multiplicity is to observe that the spectral decomposition (or Clebsch-Gordan series) stabilizes for k sufficiently large and then apply the Weyl determinant formula for $U(k)$ for sufficiently large k. This fact is proved rigorously as a theorem in Section III. In [10] it was shown that this multiplicity (for $SU(k)$) can be computed as solutions of Diophantine equations arising from the invariants of $SU(k)$. The first part of our program which is similar to the strategy given in [10] is as follows: instead of computing the multiplicity of $(\lambda)^\infty$ in the tensor product $(\lambda_1)^\infty \otimes \cdots (\lambda_r)^\infty$ we look at what is equivalent, the multiplicity of the identity representation in the augmented tensor product $(\lambda_1)^\infty \otimes \cdots \otimes (\lambda_r)^\infty \otimes (\lambda^\vee)^\infty$ where $(\lambda^\vee)^\infty$ is the contragredient representation of $(\lambda)^\infty$. But with this approach we are facing two major difficulties. The first one pertains to the contragredient representation $(\lambda^\vee)^\infty$: as it is well known (see e.g. [7]) an irreducible $(G_\infty, V^{(\lambda)^\infty})$-module can be realized as a subspace of a generalized Bargmann-Segal-Fock space in $n \times \infty$ complex variables (see Section II for this realization), but it is not known

whether the irreducible $(G_\infty, V^{(\lambda^\vee)^\infty})$-module is realizable likewise. We prove in Section V that by 'twisting' the action of the contragredient representation and by using an appropriate embedding of Bargmann-Segal-Fock spaces $\mathcal{F}_{n\times k} \subset \mathcal{F}_{n\times(k+1)} \subset \cdots$ the $(G_\infty, V^{(\lambda^\vee)^\infty})$-module can also be realized as a submodule of a Bargmann-Segal-Fock space $\mathcal{F}_{n\times\infty}$. The notable difference is that the signature of $(\lambda^\vee)^\infty$ is characterized by the *lowest weight* instead of the highest weight and we will be dealing with *lowest weight vectors* instead of highest weight vectors as in the case $(G_\infty, V^{(\lambda)^\infty})$. Another difficulty is that, realized as a submodule of the Bargmann-Segal-Fock space $\mathcal{F}_{n\times\infty}$, it is not clear that the tensor product $(\lambda_1)^\infty \otimes \cdots \otimes (\lambda_r)^\infty \otimes (\lambda^\vee)^\infty$ considered as a G_∞-module is a holomorphic representation; in particular, the identity representation might not occur in this tensor product. Using a general reciprocity theorem for holomorphic representations of some infinite-dimensional groups (see [11]) we show that the tensor product $(\lambda_1)^\infty \otimes \cdots \otimes (\lambda_r)^\infty \otimes (\lambda^\vee)^\infty$ is indeed a holomorphic representation and that the multiplicity of the identity representation of G_∞ in this augmented tensor product is indeed equal to the multiplicity of $(\lambda)^\infty$ in the tensor product $(\lambda_1)^\infty \otimes \cdots (\lambda_r)^\infty$. Having overcome this difficulty first, we still have to deal with a second major difficulty; the generators of $SU(k)$ used in [10] which are determinants of matrices of order k become unmanageable when k is large; furthermore, at the limit as $k \longrightarrow \infty$ these determinants are certainly not members of $\mathcal{F}_{n\times\infty}$. Both of these problems can be dealt with as follows: instead of using the determinant-invariants of $SU(k)$ we use the classical invariants of $U(k)$

which are generated by a system of algebraically independent polynomials, but more importantly, the number of these polynomials depends only on the tensor product $(\lambda_1)_k \otimes \cdots \otimes (\lambda_r)_k \otimes (\lambda^\vee)_k$ and not on k; in fact, the problems considered in [10] can be entirely solved using this new approach. Next, it can be shown (see [12]) that when $k \longrightarrow \infty$ these invariants tend to their *inverse* or *projective limits* which are infinite formal series of complex variables, but nevertheless remain algebraically independent and generate all $G_\infty^{\mathbb{C}}$ (or G_∞) invariants. By analogy with the definition of *rigged Hilbert Spaces* (c.f. e.g. "Generalized functions" by I. M. Gelfand and G. E. Shilov, Vol 4, P. 106) these infinite formal series may be thought of as differential operators. Thus if $f \in \mathcal{F}_{n\times\infty}$ and p is a G_∞-invariant then the inner product

$$\langle p, f\rangle = p(D)\overline{f(\bar{Z})}|_{Z=0}$$

makes perfect sense since $f \in \mathcal{F}_{n\times k}$ for some k and those terms in $p(D)$ whose column indices are larger than k simply evaluate to zero. With this new interpretation of the G_∞-invariants the method of computing Clebsch-Gordan and Racah coefficients in [10] can be adapted to the case of tensor products of G_∞; Actually, both the Diophantine equations and the computations of Clebsch-Gordan coefficients are much simpler, since the G_∞-invariants are much simpler.

II Preliminaries

Let $\mathbb{C}^{n\times k}$ denote the vector space of n row by k column matrices over $\mathbb{C}$, the field of complex numbers. If $Z = (Z_{ij})$ is an element of $\mathbb{C}^{n\times k}$, we let $\bar{Z}$ denote its complex conjugate, and write

$$Z = X_{ij} + \sqrt{-1}\; Y_{ij}\,;\;\; 1 \leq i \leq n,\; 1 \leq j \leq k.$$

If dX_{ij} (resp. dY_{ij}) denotes the Lebesgue measure on $\mathbb{R}$ we let $dZ = \prod_{1\leq i\leq n,\; 1\leq j\leq k} dX_{ij}\, dY_{ij}$ denote the Lebesgue product measure on $\mathbb{R}^{nk}$. Define a Gaussian measure $d\mu$ on $\mathbb{C}^{n\times k}$ by

$$d\mu(Z) = \pi^{-nk} \exp[tr(Z\bar{Z}^t)]dZ$$

where tr denotes the trace of a matrix. A map $f : \mathbb{C}^{n\times k} \to \mathbb{C}$ is said to be *holomorphic square integrable* if it is holomorphic on the entire domain $\mathbb{C}^{n\times k}$ and if

$$\int_{\mathbb{C}^{n\times k}} |f(Z)|^2 d\mu(Z) < \infty.$$

The holomorphic square integrable functions form a Hilbert space with respect to the inner product

$$(f_1, f_2) = \int_{\mathbb{C}^{n\times k}} f_1(Z)\overline{f_2(Z)}d\mu(Z), \tag{1}$$

of which the polynomial functions form a dense subspace. The inner product (1) is equivalent to the inner product

$$\langle f_1, f_2\rangle = f_1(D)\overline{f_2(\bar{Z})}|_{Z=0} \tag{2}$$

where $f(D)$ is the differential operator obtained by formally replacing Z_{ij} by the partial derivative $\partial/\partial Z_{ij}$. We denote this Hilbert space by $\mathcal{F}_k = \mathcal{F}(\mathbb{C}^{n\times k})$. The natural embedding of $\mathbb{C}^{n\times k}$ into $\mathbb{C}^{n\times(k+1)}$ given by

$$Z \mapsto \begin{pmatrix} & & 0 \\ & Z & \vdots \\ & & 0 \end{pmatrix} \in \mathbb{C}^{n\times(k+1)}$$

induces an isometric embedding

$$i^k_{k+1} : \mathcal{F}_k \longrightarrow \mathcal{F}_{k+1}$$

so that the collection $\{\mathcal{F}_k, i^k_{k+1}\}$ forms a directed system. We can then take the *inductive limit* $\mathcal{F}_\infty = \overline{\varinjlim \mathcal{F}_k}$ (where the bar indicates closure with respect to the norm), with the natural inclusion

$$i_k : \mathcal{F}_k \longrightarrow \mathcal{F}_\infty.$$

Formally, elements of $\mathcal{F}_\infty$ are realized as equivalence classes $[f_\alpha]$, where

$$f_\alpha \sim f_\beta \quad \text{whenever} f_\beta = i^\alpha_\beta(f_\alpha) \quad \text{and} \quad \alpha \le \beta, \quad \alpha, \beta \in \mathbb{N}.$$

Since in our case we have $\mathcal{F}_k \subset \mathcal{F}_{k+1}$, we can realize this space as

$$\mathcal{F}_\infty = \overline{\bigcup_{k=1}^{\infty} \mathcal{F}_k}.$$

If $G_k = U(k)$ (or $U(k)^{\mathbb{C}} = GL(k, \mathbb{C})$) we also have the natural inclusion

$$j^k_{k+1} : G_k \longrightarrow G_{k+1}$$

given by

$$g \mapsto \begin{pmatrix} & & & 0 \\ & g & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \in G_{k+1} \tag{3}$$

We can then take the inductive limit $G_\infty = \varinjlim G_k$, with the natural inclusion $j_k : G_k \longrightarrow G_\infty$. Again elements of G_∞ are formally defined as equivalence classes $[g_k]$, where we identify some $g_k \in G_k$ with its inclusions into G_{k+1}, G_{k+2}, etc. If we let R_k denote the representation of G_k on $\mathcal{F}_k$ given by right translation

$$R_k(g)f(Z) = f(Zg), \quad Z \in \mathbb{C}^{n\times k}, \; g \in G_k.$$

Then the following diagram commutes

$$\begin{array}{ccc} G_k \times \mathcal{F}_k & \xrightarrow{R_k} & \mathcal{F}_k \\ {\scriptstyle j^k_{k+1}\times i^k_{k+1}}\downarrow & & \downarrow{\scriptstyle i^k_{k+1}} \\ G_{k+1} \times \mathcal{F}_{k+1} & \xrightarrow[R_{k+1}]{} & \mathcal{F}_{k+1} \end{array} \tag{4}$$

and so the representation $R = \varinjlim R_k$ of G_∞ on $\mathcal{F}_\infty$ is well defined by

$$R([g_k])[f_k] = [R_k(g_k)f_k], \tag{5}$$

called the *inductive limit* of the representations R_k on $\mathcal{F}_\infty$, and we have commutativity of the diagram

$$\begin{array}{ccc} G_k \times \mathcal{F}_k & \xrightarrow{R_k} & \mathcal{F}_k \\ {\scriptstyle j_k\times i_k}\downarrow & & \downarrow{\scriptstyle i_k} \\ G_\infty \times \mathcal{F}_\infty & \xrightarrow[R]{} & \mathcal{F}_\infty \end{array}$$

Let $\mathbf{D}_k \subset G_k$ be the diagonal subgroup. Let ${\mathbf{Z}^+}_k \subset G_k$ be the unipotent subgroup of upper triangular matrices with ones along the main diagonal and let ${\mathbf{Z}^-}_k$ be the analogous lower triangular subgroup. If $(M) = (M_1, \dots, M_k)$ is any collection of integers, we define a holomorphic character

$$\pi^{(M)}(d) = d_{11}^{M_1} d_{22}^{M_2} \dots d_{kk}^{M_k} \quad d \in \mathbf{D}_k.$$

In this context an element $f \in \mathcal{F}_k$ is said to be a *weight vector* of the representation R_k with weight (M) if

$$[R_k(d)f](Z) = f(Zd) = \pi^{(M)}(d)f(Z), \qquad \forall\ d \in \mathbf{D}_k.$$

If f is a weight vector, and if

$$[R_k(\zeta)f](Z) = f(Z\zeta) = f(Z), \qquad \forall\ \zeta \in {\mathbf{Z}^+}_k$$

then $f \in \mathcal{F}_k$ is a said to be a *highest weight vector* of the representation R_k. Similarly if f is a weight vector, and if

$$[R_k(\zeta)f](Z) = f(Z\zeta) = f(Z), \qquad \forall\ \ \zeta \in {\mathbf{Z}^-}_k$$

then $f \in \mathcal{F}_k$ is a said to be a *lowest weight vector* of the representation R_k. Since $G_k = U(k)$ or $GL(k, \mathbb{C})$, each irreducible representation of G_k in $\mathcal{F}_k$ is finite dimensional and so admits a "unique" (up to multiplication by a nonzero scalar) highest weight vector with highest weight $(m) = (m_1, m_2, \dots, m_k)$, and a unique lowest weight vector with lowest weight $(m_k, m_{k-1}, \dots, m_1)$. This highest (or lowest) weight then characterizes each irreducible representation of G_k, and is called the *signature* of the representation. By the

Borel-Weil theorem a necessary and sufficient condition for (m) to be the highest weight of an irreducible polynomial representation of G_k on $\mathcal{F}_k$ is that $m_1 \geq m_2 \geq \ldots \geq m_k \geq 0$. $V^{(m)}$ is then cyclically generated as a G_k-module by the action of G_k on any one of its elements, in particular on its highest (or lowest) weight vector. Let $B_k \subset GL(k, \mathbb{C})$ be the Borel subgroup of lower triangular matrices and for a k-tuple of non-negative integers $(m) = (m_1, m_2, \ldots, m_k)$ we define a holomorphic character

$$\pi^{(m)}(b) = b_{11}^{m_1} b_{22}^{m_2} \ldots b_{kk}^{m_k} \quad b \in B_k. \tag{6}$$

As a consequence of the Borel-Weil theorem (see for example [13]), any irreducible holomorphic representation of G_k with signature $(m) = (m_1, m_2, \ldots, m_n)$ can be explicitly realized as the representation R_k on the subspace of polynomial functions in $\mathcal{F}_k = \mathcal{F}(\mathbb{C}^{n\times k})$ which satisfy the covariant condition

$$f(bZ) = \pi^{(m)}(b) f(Z), \qquad b \in B_n. \tag{7}$$

We denote this subspace by $V_k^{(m)}$, the restriction of R_k to this subspace by $R_k^{(m)}$, and where necessary we explicitly designate this irreducible representation by the pair $(R_k^{(m)}, V_k^{(m)})$.

For each $k = 1, 2, \ldots$, let V_k be a subspace of $\mathcal{F}_k$, on which the representation R_k of G_k is irreducible. Suppose also that the following diagram commutes

$$\begin{array}{ccc} G_k \times V_k & \xrightarrow{R_k} & V_k \\ {\scriptstyle j_{k+1}^k \times i_{k+1}^k}\downarrow & & \downarrow{\scriptstyle i_{k+1}^k} \\ G_{k+1} \times V_{k+1} & \xrightarrow[R_{k+1}]{} & V_{k+1} \end{array} \tag{8}$$

or equivalently, that the restriction of R_{k+1} to G_k contains a representation equivalent to R_k. In this case we write $V_k \preceq V_{k+1}$, and it is well documented in the literature that the representation $R = \varinjlim R_k$ of G_∞ on $V = \varinjlim V_k$ is also irreducible. (For detailed expositions of inductive limit representations see [8] [2] [5].)

If V_k is an irreducible representation of G_k with signature

$$(m) = (m_1, m_2, \dots, m_k) \text{ with } m_1 \geq m_2 \geq \dots \geq m_k \geq 0,$$

and if V_{k+1} is an irreducible representation of G_{k+1} with signature $(h) = (h_1, h_2, \dots, h_n, h_{n+1})$ it is also well known that $V_k \preceq V_{k+1}$ or equivalently written $(m) \preceq (h)$ if and only if

$$h_i \geq m_i \geq h_{i+1}, \quad i = 1, \dots, k.$$

In particular, if $(m_1, m_2, \dots, m_k)$ is the signature of an irreducible representation of G_k and $(m_1, m_2, \dots, m_k, 0)$ is the signature of an irreducible representation of G_{k+1}, then

$$(m_1, m_2, \dots, m_k) \preceq (m_1, m_2, \dots, m_k, 0)$$

and it is easy to see that if f_{max} is a highest weight vector for an irreducible representation of G_k with highest weight $(m_1, m_2, \dots, m_k)$ then f_{max} is also a highest weight vector of the irreducible representation of G_{k+1} with highest weight $(m_1, m_2, \dots, m_k, 0)$. We denote the inductive limit of the representations

$$(m_1, m_2, \dots, m_k) \preceq (m_1, m_2, \dots, m_k, 0) \preceq (m_1, m_2, \dots, m_k, 0, 0) \preceq \cdots$$

by

$$(m_1, m_2, \dots, m_k, 0, \dots) = (m_1, m_2, \dots, m_k, \overrightarrow{0}) = (m)^\infty$$

and realize this representation as the submodule of $\mathcal{F}_\infty$ generated by the action of G_∞ on this highest weight vector. In the sequel we may also require more explicit notation: If $\underbrace{(m_1, m_2, \dots, m_l, 0, \dots, 0)}_{k}$ is the signature of an irreducible representation of G_k call the integers $m_1, m_2, \dots, m_l, 0$ the *entries*, we say l is the *length* of the signature (i.e., if the signature has at most l non-zero entries) and write

$$\underbrace{(m_1, m_2, \dots, m_l, 0, \dots, 0)}_{k} = (m)_l^k$$

or just $(m)^k$ if it is unnecessary to specify the length. With this notation we denote the signature of the inductive limit of the representations

$$(m_1, m_2, \dots, m_l) \preceq (m_1, m_2, \dots, m_l, 0) \preceq (m_1, m_2, \dots, m_l, 0, 0) \preceq \cdots$$

$$= (m)_l^k \preceq (m)_l^{k+1} \preceq (m)_l^{k+2} \preceq \cdots$$

by

$$(m)_l^\infty = (m_1, m_2, \dots, m_l, 0, 0, 0, \dots) = (m_1, m_2, \dots, m_l, \overrightarrow{0}).$$

III Stability of spectral decompositions

We motivate this section with the following example. It is readily computed using one of the standard formulae (for example [13]) that the tensor product of the irreducible representations of G_2 ($= U(2)$ or $GL(2,\mathbb{C})$) decomposes as a direct sum

$$\begin{aligned}&(1,0)\otimes(2,0)\otimes(2,0)\otimes(3,0)\\&\qquad = (8,0)+3(7,1)+5(6,2)+5(5,3)+2(4,4) \quad (9)\end{aligned}$$

and the tensor product of irreducible representations of G_4

$$\begin{aligned}&(1,0,0,0)\otimes(2,0,0,0)\otimes(2,0,0,0)\otimes(3,0,0,0)\\&\quad = (8,0,0,0)+3(7,1,0,0)+5(6,2,0,0)+5(5,3,0,0)+2(4,4,0,0)\\&\quad + 3(6,1,1,0)+6(5,2,1,0)+5(4,3,1,0)+3(4,2,2,0)+2(3,3,2,0)\\&\quad + (5,1,1,1)+2(4,2,1,1)+(3,3,1,1)+(3,2,2,1) \quad (10)\end{aligned}$$

But notice that the first line of (10) is just (9), the spectrum of G_2 embedded in the spectrum of G_4. In this case we say that the spectrum of G_4 *contains* the spectrum of G_2, or that the spectrum of G_2 *appears in* the spectrum of G_4. Furthermore, it is routine to check that the spectral decomposition of

irreducible representations of G_5 is given by

$$\begin{aligned}
&(1,0,0,0,0)\otimes(2,0,0,0,0)\otimes(2,0,0,0,0)\otimes(3,0,0,0,0)\\
&=(8,0,0,0,0)+3(7,1,0,0,0)+5(6,2,0,0,0)+5(5,3,0,0,0)+2(4,4,0,0,0)\\
&\quad+3(6,1,1,0,0)+6(5,2,1,0,0)+5(4,3,1,0,0)+3(4,2,2,0,0)\\
&\quad+2(3,3,2,0,0)+(5,1,1,1,0)+2(4,2,1,1,0)+(3,3,1,1,0)\\
&\quad+(3,2,2,1,0)
\end{aligned} \tag{11}$$

and that the corresponding spectral decompositions of $G_6,\ G_7,\ldots$ are the same, i.e. composed entirely of the embedding of the spectrum of G_4. In this case we say the spectral decomposition *stabilizes.*

Proposition III.1. *If $(\alpha)_k^k=(\alpha_1,\alpha_2,\ldots,\alpha_k)$ and $(\beta)_k^k=(\beta_1,\ldots,\beta_k)$ are the signatures of irreducible representations of G_k, and if $(\alpha)_k^{k+1}=(\alpha_1,\alpha_2,\ldots,\alpha_k,0)$ and $(\beta)_k^{k+1}=(\beta_1,\ldots,\beta_k,0)$ are the signatures of irreducible representations of G_{k+1}, then the spectrum of $(\alpha)_k^k\otimes(\beta)_k^k$ appears in the spectrum of $(\alpha)_k^{k+1}\otimes(\beta)_k^{k+1}$. Furthermore, the spectrum of $(\alpha)_k^K\otimes(\beta)_k^K$ stabilizes for K sufficiently large.*

Proof. We first note that, in the special case where $(\alpha)_1^k=\underbrace{(\alpha_1,0\ldots,0)}_{k}$ by [13] the spectral decomposition of $(\alpha)_1^k\otimes(\beta)_k^k$ is given by the 'Weyl formula', which is equivalent to applying the multiplier Γ_{α_1} to the signature $(\beta)_k^k=(\beta_1,\ldots,\beta_k)$ where

$$\Gamma_{\alpha_1}(\beta_1,\ldots,\beta_k)=\sum_{\substack{\nu_1+\ldots+\nu_k=\alpha_1\\0\le\nu_{i+1}\le s_i}}(\beta_1+\nu_1,\ldots,\beta_k+\nu_k)$$

Here, and in what follows of this proof, the Weyl formula also requires the condition that $0 \leq \nu_{i+1} \leq s_i$ where $s_i = m_i - m_{i+1}$, and we will refer to a multiplier of this type as a *simple multiplier*.

Now applying this simple multiplier to the signature $(\beta)_k^{k+1}$ we have

$$\Gamma_{\alpha_1}(\beta_1, \ldots, \beta_k, 0) = \sum_{\substack{\nu_1+\ldots+\nu_k+\nu_{k+1}=\alpha_1 \\ 0\leq\nu_{i+1}\leq s_i}} (\beta_1 + \nu_1, \ldots, \beta_k + \nu_k, 0 + \nu_{k+1})$$

$$= \sum_{\substack{\nu_1+\ldots+\nu_{k+1}=\alpha_1 \\ \nu_{k+1}=0}} (\beta_1 + \nu_1, \ldots, \beta_k + \nu_k, 0 + \nu_{k+1}) + \sum_{\substack{\nu_1+\ldots+\nu_{k+1}=\alpha_1 \\ \nu_{k+1}\neq 0}} (\beta_1 + \nu_1, \ldots, \beta_k + \nu_k, 0 + \nu_{k+1})$$

But the first sum, with $\nu_{k+1} = 0$, is just the spectrum of $(\alpha)_1^k \otimes (\beta)_k^k$ contained in the spectrum of $(\alpha)_1^{k+1} \otimes (\beta)_k^{k+1}$

We next note that a similar situation occurs when we apply a second simple multiplier Γ_{α_2} to the above spectral decomposition $(\alpha)_1^{k+1} \otimes (\beta)_k^{k+1}$. That is, the sums are grouped into those terms whose last entry is zero, and those terms whose last entry is non-zero;

$$\Gamma_{\alpha_2}\left[(\alpha)_1^{k+1} \otimes (\beta)_k^{k+1}\right] = \Gamma_{\alpha_2}(\Gamma_{\alpha_1}(\beta_1, \ldots, \beta_k, 0))$$

$$= \Gamma_{\alpha_2} \left[\sum_{\substack{\nu_1+\ldots+\nu_k+\nu_{k+1}=\alpha_1 \\ \nu_{k+1}=0}} (\beta_1+\nu_1,\ldots,\beta_k+\nu_k, 0+\nu_{k+1}) \right.$$

$$\left. + \sum_{\substack{\nu_1+\ldots+\nu_k+\nu_{k+1}=\alpha_1 \\ \nu_{k+1}\neq 0}} (\beta_1+\nu_1,\ldots,\beta_k+\nu_k, 0+\nu_{k+1}) \right]$$

$$= \sum_{\substack{\nu_1+\ldots+\nu_k+\nu_{k+1}=\alpha_1 \\ \nu_{k+1}=0}} \Gamma_{\alpha_2}(\beta_1+\nu_1,\ldots,\beta_k+\nu_k, 0)$$

$$+ \sum_{\substack{\nu_1+\ldots+\nu_k+\nu_{k+1}=\alpha_1 \\ \nu_{k+1}\neq 0}} \Gamma_{\alpha_2}(\beta_1+\nu_1,\ldots,\beta_k+\nu_k, 0+\nu_{k+1})$$

$$= \sum_{\substack{\nu_1+\ldots+\nu_{k+1}=\alpha_1 \\ \nu_{k+1}=0}} \sum_{\substack{\mu_1+\ldots+\mu_{n+1}=\alpha_2 \\ \mu_{k+1}=0}} (\beta_1+\nu_1+\mu_1,\ldots,\beta_k+\nu_k+\mu_k, 0)$$

$$+ \sum (\text{other terms involving signatures whose last entry is non-zero})$$

We then extend this idea to the general case where the spectral decomposition of $(\alpha)^k_k \otimes (\beta)^k_k$ is given by applying the multiplier $\Gamma_{\alpha^k_k}$ to the signature β^k_k where $\Gamma_{\alpha^k_k}$ is a *compound multiplier* computed as the 'Weyl Determinant' [13];

$$\Gamma_{\alpha^k_k} = \begin{vmatrix} \Gamma_{\alpha_1} & \Gamma_{\alpha_1+1} & \cdots & \Gamma_{\alpha_1+(k-1)} \\ \Gamma_{\alpha_2-1} & \Gamma_{\alpha_2} & \cdots & \Gamma_{\alpha_2+(k-2)} \\ \vdots & \cdots & & \vdots \\ \Gamma_{\alpha_k-(k-1)} & \Gamma_{\alpha_k-(k-2)} & \cdots & \Gamma_{\alpha_k} \end{vmatrix}$$

Here the simple multipliers Γ_α are regarded as permutable operators and the determinant is expanded in the usual way, with $\Gamma_0 = 1$ and $\Gamma_\alpha = 0$ for $\alpha < 0$. From this last statement it is obvious that the compound multiplier $\Gamma_{\alpha_k^{k+1}}$ is equal to $\Gamma_{\alpha_k^k}$ since the $k+1^{st}$ row used to compute the determinant corresponding to $\Gamma_{\alpha_k^{k+1}}$is just $(0, \ldots, 0, 1)$.

Now for notational convenience we set the simple multiplier $\Gamma_{i,j} = \Gamma_{\alpha_i-(i-j)}$ and using the usual formula for determinant (summing over S_k, the symmetric group on k symbols) we have

$$\Gamma_{\alpha_k^{k+1}} = \Gamma_{\alpha_k^k} = \sum_{\sigma \in S_k} sgn(\sigma)\Gamma_{\alpha_1\sigma(1)} \cdots \Gamma_{\alpha_k\sigma(k)}$$

So

$$\Gamma_{\alpha_k^{k+1}}(\beta_k^{k+1})$$

$$= \sum_{\sigma \in S_n} sgn(\sigma)\Gamma_{\alpha_1\sigma(1)} \cdots \Gamma_{\alpha_k\sigma(k)}(\beta_1, \ldots, \beta_k, 0)$$

$$= \sum_{\sigma \in S_k} sgn(\sigma) \underbrace{\sum_{\nu_1+\ldots+\nu_{k+1}=\alpha_1} \cdots \sum_{\mu_1+\ldots+\mu_{k+1}=\alpha_k}}_{k \text{ sums}} (\beta_1+\nu_1+\cdots+\mu_1, \ldots, 0+\nu_{k+1}+\cdots+\mu_{k+1}) \tag{12}$$

$$= \sum_{\sigma \in S_k} sgn(\sigma) \sum \cdots \sum (\text{signatures whose last entry is zero}) \tag{13}$$

$$+ \sum_{\sigma \in S_k} sgn(\sigma) \sum \cdots \sum (\text{signatures whose last entry is non-zero})$$

But the sum (13) is just the spectrum of $(\alpha)_k^k \otimes (\beta)_k^k$ appearing in the spectrum of $(\alpha)_k^{k+1} \otimes (\beta)_k^{k+1}$. Finally, the requirement that $0 \leq \nu_{i+1} \leq m_i - m_{i+1}$ guarantees that the application of a simple multiplier to a signature $(m_1, \ldots, m_l, 0, \ldots, 0)$ extends the length of the signature by at most one, since $0 \leq \nu_{l+1} \leq (m_{l+1} - m_{l+2}) = 0$. Thus, since there are only k sums in (12), application of a compound multiplier corresponding to a signature of length k decomposes the tensor product into a spectrum of signatures of length at most $l + k$, proving that the spectrum stabilizes.

$\square$

IV A reciprocity theorem

According to [11] we have the following theorem regarding *dual representations* of Bargmann-Segal-Fock spaces.

Theorem IV.1. *Let*

$$G_1 \subset G_2 \subset \cdots \subset G_k \subset G_{k+1} \subset \cdots$$

be a chain of compact classical groups. Let G_∞ denote the inductive limit of the G_k's. Let R_{G_∞} and $R'_{G'}$ be given dual representations on $\mathcal{F}_{n\times\infty}$. Let H_∞ be the inductive limit of a chain of compact subgroups

$$H_1 \subset H_2 \subset \cdots \subset H_k \subset H_{k+1} \subset \cdots$$

with $H_k \subset G_k$, and let R_{H_∞} be the representation of H_∞ on $\mathcal{F}_{n\times\infty}$ obtained by restricting R_{G_∞} to H_∞. If there exists a group $H' \supset G'$ and a representation

$R'_{H'}$ on $\mathcal{F}_{n\times\infty}$ such that $R'_{H'}$ is the dual to R_{H_∞} and $R'_{G'}$ is the restriction of $R'_{H'}$ to the subgroup G' of H' then we have the following multiplicity free decompositions of $\mathcal{F}_{n\times\infty}$ into isotypic components

$$\mathcal{F}_{n\times\infty} = \sum_{(\lambda)} \oplus \mathcal{I}^{(\lambda)}_{n\times\infty} = \sum_{(\mu)} \oplus \mathcal{I}^{(\mu)}_{n\times\infty}$$

where (λ) is a common irreducible signature of the pair (G', G_∞) and (μ) is a common signature of the pair (H', H_∞).

If λ_{G_∞} (resp. $\lambda'_{G'}$) denotes an irreducible unitary representation of class (λ) and μ_{H_∞} (resp. $\mu'_{H'}$) denotes an irreducible unitary representation of class

(μ), then the multiplicity

$$\dim\left[\mathrm{Hom}_{H_\infty}(\mu_{H_\infty} : \lambda_{G_\infty|_{H_\infty})})\right]$$

of the irreducible representation μ_{H_∞} in the restriction to H_∞ of the representation λ_{G_∞} is equal to the multiplicity

$$\dim\left[\mathrm{Hom}_{G'}(\lambda'_{G'} : \mu'_{H'|_{G'}})\right]$$

of the irreducible representation $\lambda'_{G'}$ in the restriction of the representation $\mu'_{H'}$.

Note that G' and H' are finite dimensional Lie groups and that we have a similar theorem for the pairs (G', G_k) and (H', H_k) where G' and H' remain fixed for all k. It follows that

$$\dim\left[\mathrm{Hom}_{H_\infty}(\mu_{H_\infty} : \lambda_{G_\infty|_{H_\infty}})\right]$$

remains constant and the spectral decomposition of $\lambda_{G_k}\big|_{H_k}$ stabilizes for k large. To apply this theorem to our problem we first let

$$G_\infty = \underbrace{U_\infty \times \cdots \times U_\infty}_{r \ copies}$$

acting as exterior tensor product representations on

$$V^{(m_1)^\infty} \otimes \cdots \otimes V^{(m_r)^\infty} \subset \mathcal{F}_{n\times\infty},$$

and $G' = U(p_1) \times \cdots U(p_r)$ acting on $\mathcal{F}_{n\times\infty}$.

Then $H_\infty = U_\infty$ is the interior tensor product representation on

$$V^{(m_1)^\infty} \otimes \cdots \otimes V^{(m_r)^\infty},$$

and $H' = U(n)$, where recall that $n = p_1 + p_2 + \cdots + p_r$. This gives the multiplicity of the representation of U_∞ with signature $(m)^\infty$ in $(m_1)^\infty \otimes \cdots \otimes (m_r)^\infty$ in terms of the multiplicity of the representation of the corresponding representations of $U(p_1) \times \cdots \times U(p_r)$ in the corresponding representation on $U(n)$. Next we let

$$G_\infty = \underbrace{U_\infty \times \cdots \times U_\infty}_{r+1 \ copies}$$

acting as exterior tensor product representations on

$$V^{(m_1)^\infty} \otimes \cdots \otimes V^{(m_r)^\infty} \otimes V^{(m^\vee)^\infty} \subset \mathcal{F}_{n\times\infty}$$

and $G' = U(p_1) \times \cdots \times U(p_r) \times U(q)$ acting on $\mathcal{F}_{n\times\infty}$. Then $H_\infty = U_\infty$ and $H' = U(p,q)$ where $p+q = n,\ p_1 + \cdots + p_r = p$. This gives the multiplicity of

the representation of U_∞ with signature $(0,\dots,0,\dots)^\infty$ in the tensor product $(m_1)^\infty \otimes\cdots\otimes (m_r)^\infty \otimes (m^\vee)^\infty$ in terms of the multiplicity of the representation with signature $(m_1)^n \otimes \cdots \otimes (m_r)^n \otimes (m^\vee)^n$ of $U(p_1) \times \cdots \times U(p_r) \times U(q)$ in the holomorphic discrete series of $U(p,q)$ with signature (lowest highest weight) $(\underbrace{0,\dots,0}_{p},\underbrace{0,\dots,0}_{q})$. Note that these two applications of this theorem can be used together to give another proof of Theorem VI.1 of Section VI.

V Realization of the contragredient representation

A representation ρ of any group G on a vector space V induces in a natural way a representation ρ^* (said to be *contragredient* to ρ) on its dual space V^* by

$$\rho^*(g)\phi(v) = \phi(\rho(g^{-1})v) \quad \phi \in V^*,\ g \in G.$$

In this section, by making a formal change of variable, we are able to realize R_k^* (the representation contragredient to R_k) as the representation $R_k^\vee$ on a subspace of polynomial functions of the Fock space $\mathcal{F}_k^\vee$.

Let $\langle\ \ |\ \ \rangle$ be the inner product on the space $\mathcal{F}_k$ given by (2) or the equivalent inner product (1). Then for any $f \in \mathcal{F}_k$ and for each $k = 1,2,3,\dots$ the mapping

$$\Phi : \mathcal{F}_k \longrightarrow \mathcal{F}_k^*$$

given by

$$[\Phi f](h) = \langle h|f\rangle \qquad h \in \mathcal{F}_k$$

is a conjugate linear isomorphism (or *anti-isomorphism*) from $\mathcal{F}_k$ onto its dual space $\mathcal{F}_k^*$, and it is routine to check (see [10]) that Φ intertwines the representations R_k and R_k^*. It follows that if $(R_k^{(m)}, V_k^{(m)})$ is an irreducible representation of G_k, and if $V_k^{(m^*)} = \Phi(V^{(m)})$, then $(R_k^{(m^*)}, V_k^{(m^*)})$ is also an irreducible representation of G_k. It is shown in appendix A of [10] that the highest weight vector of $(R_k^{(m)}, V_k^{(m)})$ with highest weight $(m_1, m_2, \dots, m_k)$ is mapped to the lowest weight vector of $(R_k^{(m^*)}, V_k^{(m^*)})$ with weight $(-m_1, -m_2, \dots, -m_k)$, and the lowest weight vector of $(R_k^{(m)}, V_k^{(m)})$ with weight $(m_k, \dots, m_1)$ is mapped onto the highest weight vector of $(R_k^{(m^*)}, V_k^{(m^*)})$ with weight $(-m_k, \dots, -m_1)$. We will realize $(R_k^{(m^*)}, V_k^{(m^*)})$ on a Fock space $\mathcal{F}_k^\vee$ as the representation $(R_k^{(m)^\vee}, V^{(m^\vee)})$ constructed as follows.

Define $(\mathbb{C}^{n\times k})^\vee$ as the vector space of complex $n \times k$ matrices with the reverse ordering

$$w = \begin{pmatrix} w_{n,k} & \cdots & w_{n,1} \\ \vdots & & \vdots \\ w_{1,k} & \cdots & w_{1,1} \end{pmatrix} \in (\mathbb{C}^{n\times k})^\vee$$

Let $s = s_k$ be the $k \times k$ matrix with ones along the off diagonal and zeros elsewhere

$$
s = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & & & \cdot & 0 \\ \vdots & & \cdot & & \vdots \\ 0 & \cdot & & & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathbb{C}^{k\times k} \tag{14}
$$

For future reference one easily checks that if we set

$$
Z = \begin{pmatrix} w_{11} & \cdots & w_{1k} \\ \vdots & & \vdots \\ w_{n1} & \cdots & w_{nk} \end{pmatrix}
$$

that $W = s_n Z s_k$ and that $s = s^{-1} = s^T$. We then let $\mathcal{F}_k^\vee$ be the Hilbert space of holomorphic square-integrable functions on $(\mathbb{C}^{n\times k})^{\ \vee}$ and define the action $R_k^\vee$ of G_k on $\mathcal{F}_k^\vee$ as

$$
[R_k^\vee(g)f](w) = f(w(sgs)^\vee) = f(w(sg^\vee s)).
$$

The embedding of $(\mathbb{C}^{n\times k})^{\ \vee}$ into $(\mathbb{C}^{n\times(k+1)})^{\ \vee}$ given by

$$
W \mapsto \begin{pmatrix} 0 & \\ \vdots & W \\ 0 & \end{pmatrix} \in \mathbb{C}^{n\times(k+1)\ \vee}
$$

then induces an embedding $i_{k+1}^{k\ \vee} : \mathcal{F}_k^\vee \longrightarrow \mathcal{F}_{k+1}^\vee$, we have commutativity of the diagram

$$
\begin{array}{ccc}
G_k \times \mathcal{F}_k^\vee & \xrightarrow{R_k^\vee} & \mathcal{F}_k^\vee \\
{\scriptstyle j_{k+1}^k \times i_{k+1}^{k\ \vee}} \downarrow & & \downarrow {\scriptstyle i_{k+1}^{k\ \vee}} \\
G_{k+1} \times \mathcal{F}_{k+1}^\vee & \xrightarrow[R_{k+1}^\vee]{} & \mathcal{F}_{k+1}^\vee
\end{array}
$$

as in (4), and the inductive limit representation $R^\vee = \varinjlim R_k^\vee$ of G on $\mathcal{F}_\infty^\vee = \varinjlim \mathcal{F}_k^\vee$ is well defined as in (5).

We remark here that the space $\mathcal{F}_\infty^\vee$ defined above and the space $\mathcal{F}_\infty$ defined in Section II, are certainly equal *as sets*, but are somewhat different as algebraic objects, being induced by different embeddings. In what follows, the arguments presented in developing properties of the various inductive limit representations are readily modified to any situation.

Let f_{max} be the highest weight vector of $(R_k^{(m)}, V_k^{(m)})$ with highest weight $(m_1, m_2, \dots, m_k)$. Then by definition

$$R(d)f_{max}(Z) = d_{11}^{m_1} \cdots d_{kk}^{m_k} f_{max}(Z) = \pi^{(m_1, m_2, \dots, m_k)}(d)f_{max}(Z).$$

Define $f_{min}^{\vee}(W) := f_{max}(s_n W s_k) = f_{max}(Z)$, so that $f_{max}(Z) = f_{min}^{\vee}(s_n Z s_k)$. Then

$$\begin{aligned}
R_k^\vee(d)f_{min}^{\vee}(W) &= f_{min}^{\vee}(W s d^{\vee} s) \\
&= f_{max}(s_n(W \; s d^{\vee} s)s), \quad \text{since } s_n W s = Z \text{ and } d^{\vee} = d^{-1} \\
&= f_{max}(Z d^{-1}) \\
&= \pi^{(m_1, m_2, \dots, m_k)}(d^{-1})f_{max}(Z) \\
&= \pi^{(-m_1, -m_2, \dots, -m_k)}(d)f_{max}(Z) \\
&= \pi^{(-m_1, -m_2, \dots, -m_k)}(d)f_{min}^{\vee}(s_n Z s_k) \\
&= \pi^{(-m_1, -m_2, \dots, -m_k)}(d)f_{min}^{\vee}(W)
\end{aligned}$$

And for $\zeta \in \mathbf{Z}^-$

$$
\begin{aligned}
R_k^\vee(\zeta) \overset{\vee}{f}_{min}(W) &= \overset{\vee}{f}_{min}(W s \overset{\vee}{\zeta} s) \\
&= f_{max}(s_n(W\ s\overset{\vee}{\zeta} s)s) \\
&= f_{max}(Z\overset{\vee}{\zeta}) \\
&= f_{max}(Z) \\
&= \overset{\vee}{f}_{min}(W)
\end{aligned}
$$

since if $\zeta \in \mathbf{Z}^-$, then $\overset{\vee}{\zeta} \in \mathbf{Z}^+$. Thus $\overset{\vee}{f}_{min}$ is a lowest weight vector for the representation $R_k^\vee$, so if we let $V_k^{(\overset{\vee}{m})}$ be the G_k-submodule generated by the action $R_k^\vee$ on $\overset{\vee}{f}_{min}$, then $(R_k^\vee, V_k^{(\overset{\vee}{m})})$ is an irreducible representation of G_k characterized by its lowest weight $(-m_1, -m_2, \ldots, -m_k)$, and it follows that this representation is equivalent to the contragredient representation on the dual space $(R_k^{(m^*)}, V_k^{(m^*)})$, with the same lowest weight $(-m_1, -m_2, \ldots, -m_k)$. Furthermore, if $b \in B_n$, and if we set $\tilde{b} = s_n b s_n$ then, using an argument similar to the one above, we see that

$$
\overset{\vee}{f}_{min}(\tilde{b}W) = \pi^{(m)}(b)\overset{\vee}{f}_{min}(W)
$$

and thus the space $(R_k^\vee, V_k^{(\overset{\vee}{m})})$ can be characterized as the subspace of polynomial functions that transform covariantly with respect to the Borel subgroup, as in (6). We remark here that we refer to $\overset{\vee}{f}_{min}$ as a *lowest weight vector* because it is invariant under right translation by the subgroup $\mathbf{Z}^-$ which corresponds to the notion of lowest weight using the usual lexico-

graphic ordering. It is strictly a matter of choice whether or not to refer to it as a highest weight vector with respect to the reverse lexicographic ordering.

With this realization, if $(m) = (m_1, m_2, \ldots, m_k)$ is the signature of an irreducible representation of G_k, then its contragredient representation has signature $(m^\vee) = (-m_1, -m_2, \ldots, -m_k)$, and it is routine to check that

$$(-m_1, \ldots, -m_k) \prec (-m_1, \ldots, -m_k, 0) \prec (-m_1, \ldots, -m_k, 0, 0) \prec \cdots \tag{15}$$

i.e. the appropriate diagram (see (8)) commutes, and so the inductive limit of the irreducible representations (15) is an irreducible representation of G_∞ with signature

$$(m^\vee)^\infty =: (-m_1, -m_2, \ldots, -m_l, \overrightarrow{0})$$

generated by the action $R^\vee$ on the vector $\overset{\vee}{f}_{min}$. We will adopt the convention of refering to this as the representation contragredient to the irreducible representation with signature $(m)^\infty = (m_1, m_2, \ldots, m_k, \overrightarrow{0})$, although it is the **inductive limit** of contragredient representations. We summarize with

Theorem V.1. *If the irreducible representation of G_∞ with signature*

$$(m_1, m_2, \ldots, m_l, \overrightarrow{0})$$

is the inductive limit of the representations

$$(m_1, m_2, \ldots, m_l) \preceq (m_1, m_2, \ldots, m_l, 0) \preceq (m_1, m_2, \ldots, m_l, 0, 0) \preceq \cdots$$

then the inductive limit of contragredient representations

$$(m_1, m_2, \ldots, m_l)^* \preceq (m_1, m_2, \ldots, m_l, 0)^* \preceq (m_1, m_2, \ldots, m_l, 0, 0)^* \preceq \cdots$$

is an also irreducible representation of G_∞ with signature

$$(-m_1, -m_2, \dots, -m_l, \overrightarrow{0}).$$

We illustrate this idea with the following example. For each $k = 1, 2, 3, \dots$ consider $\mathcal{F}_k = \mathcal{F}(\mathbb{C}^{1\times k})$. If

$$f^{(m)}(Z) = z_1^m, \qquad Z = (z_1, z_2, \dots, z_k) \in \mathbb{C}^{1\times k}$$

it is easy to check that

$$f^{(m)}(Zd) = d_{11}^m z_1^m = \pi^{(m,0,\dots,0)}(d) f^{(m)}(Z)$$

and

$$f^{(m)}(Z\zeta) = f^{(m)}(Z), \quad \zeta \in \mathbf{Z}^+$$

so that $f^{(m)}(Z)$ is a highest weight vector of the representation, with highest weight $(\underbrace{m, 0, \dots 0}_{k})$. Right translation of $f^{(m)}$ by G_k generates the finite dimensional vector space $P^{(m)}(\mathbb{C}^{1\times k})$, of homogeneous polynomials of degree m, so that $V^{(m,0,\dots,0)} = P^{(m)}(\mathbb{C}^{1\times k})$ is an irreducible representation of G_k with signature $(\underbrace{m, 0, \dots 0}_{k})$. Now $P^{(m)}(\mathbb{C}^{1\times k})$ embeds isometrically into $P^{(m)}(\mathbb{C}^{1\times (k+1)})$, which is also generated as a G_{k+1}- module by right translation of the highest weight vector z_1^m, which now has highest weight $(\underbrace{m, 0, \dots 0}_{k+1})$. Taking the inductive limit of the irreducible representations

$$(m) \prec (m, 0) \prec \dots \prec (\underbrace{m, 0, \dots 0}_{k}) \prec (\underbrace{m, 0, \dots 0}_{k+1}) \prec \dots$$

we obtain $V^{(m,\overrightarrow{0})}$, the irreducible representation of G_∞ with signature $(m,\overrightarrow{0}) = (m,0,0,\dots)$, which is realized in $\mathcal{F}_\infty$ as the subspace of homogeneous polynomials of degree m, generated by the action R of G_∞ on the highest weight vector $f^{(m)}(Z) = z_1^m$;

$$V^{(m,\overrightarrow{0})} = P^{(m)}(z_1, z_2, \dots), \quad Z \in \mathbb{C}^{1\times\infty}$$

Now if $w = (w_k, \dots, w_2, w_1) \in \mathbb{C}^{1\times k\ \vee}$ set $f^{(-m)}(w) = w_1^m$.

If $d =$ diagonal $(d_{11}, \dots, d_{kk}) \ \in \mathbf{D}_k$, then

$$\begin{aligned} \left[R_k^\vee(d) f^{(-m)}\right](w) = & \ f^{(-m)}(w(sd^\vee s)) \\ & = (d_{11}^{-1} w_{11})^m = d_{11}^{-m} f^{(-m)}(w) = \pi^{(-m,0,\dots,0)}(d) f^{(-m)}(Z) \end{aligned}$$

and if $\zeta \in \mathbf{Z}^{-}{}_k$ then

$$\begin{aligned} \left[R_k^\vee(\zeta) f^{(-m)}\right](w) = & \ f^{(-m)}(w(s\zeta^\vee s)) \\ & = (\zeta_{11} w_{11})^m = f^{(-m)}(Z) \quad \text{since } \zeta_{11} = 1 \end{aligned}$$

Thus $f^{(-m)}$ is a lowest weight vector for the representation $R_k^\vee$ with lowest weight $\overbrace{(-m, \dots, 0)}^{k}$, and since this holds for all k, we denote the signature of the inductive limit of the representations

$$(-m) \prec (-m, 0) \prec \dots \prec (\underbrace{-m, 0, \dots 0}_{k}) \prec (\underbrace{-m, 0, \dots 0}_{k+1}) \prec \dots$$

by $(-m, 0, \dots) = (-m, \overrightarrow{0})$, and the irreducible representation $V^{(-m,\overrightarrow{0})}$ is realized as the space of homogeneous polynomials of degree m on $\mathbb{C}^{1\times\infty^\vee}$, generated by $f^{(-m)}$.

$$V^{(-m,\overrightarrow{0})} = P^{(m)}(\dots, w_2, w_1), \quad w \in \mathbb{C}^{1\times\infty^\vee}$$

VI Decomposing tensor products of irreducible representations

We now use this construction to realize the tensor product of inductive limits of irreducible representations. For

$$Z^i = \begin{pmatrix} z^i_{11} & z^i_{12} & \cdots & z^i_{1k} \\ \vdots & & & \vdots \\ z^i_{p_i1} & z^i_{p_i2} & \cdots & z^i_{p_ik} \end{pmatrix} \in \mathbb{C}^{p_i \times k}$$

set

$$\begin{pmatrix} Z \\ W \end{pmatrix} = \begin{pmatrix} Z^1 \\ Z^2 \\ \vdots \\ Z^r \\ W \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1k} \\ \vdots & & & \vdots \\ z_{p1} & z_{p2} & \cdots & z_{pk} \\ w_{qk} & \cdots & w_{q2} & w_{q1} \\ \vdots & & & \vdots \\ w_{1k} & \cdots & w_{12} & w_{11} \end{pmatrix} \in \mathbb{C}^{p\times k} \oplus \mathbb{C}^{q\times k^\vee}$$

where $p_1 + \cdots + p_r = p$ and $p + q = n$. For economy of notation, we now let $\mathcal{F}_k$ be the set of holomorphic square integrable functions on $\mathbb{C}^{p\times k} \oplus \mathbb{C}^{q\times k^\vee}$ and define a representation of G_k on $\mathcal{F}_k$ by

$$[R_k \otimes R_k^\vee(g) f]\left(\begin{pmatrix} Z \\ W \end{pmatrix}\right) = f\left(\begin{pmatrix} Z\,g \\ W\,(sg^\vee s) \end{pmatrix}\right) \tag{16}$$

We then obtain the inductive limit representation $R \otimes R^\vee$ of the group G_∞ on $\mathcal{F}_\infty = \overline{\varinjlim \mathcal{F}_k}$ as the representation induced by the embedding of

$$\mathbb{C}^{p\times k} \oplus \mathbb{C}^{q\times k^\vee} \longrightarrow \mathbb{C}^{p\times (k+1)} \oplus \mathbb{C}^{q\times (k+1)^\vee}$$

given by

$$\begin{pmatrix} Z \\ W \end{pmatrix} \mapsto \begin{pmatrix} z_{11} & z_{22} & \cdots & z_{1k} & 0 \\ \vdots & & & & \vdots \\ z_{p1} & z_{p2} & \cdots & z_{pk} & 0 \\ 0 & w_{qk} & \cdots & w_{q2} & w_{q1} \\ \vdots & & & & \vdots \\ 0 & w_{1k} & \cdots & w_{12} & w_{11} \end{pmatrix} \in \mathbb{C}^{p\times(k+1)} \oplus \mathbb{C}^{q\times(k+1)^\vee} \tag{17}$$

and the embedding of $G_k \longrightarrow G_{k+1}$ given by (3).

If $(m)^i = \overbrace{(m_1^i, m_2^i, \ldots, m_{p_i}^i, 0, \ldots, 0)}^{k}$ is the signature of an irreducible representation of G_k, and if $(m)^\vee = (-m_1, -m_2, \ldots, -m_q, 0, \ldots, 0)$ is the signature of the representation contragredient to $(m) = (m_1, m_2, \ldots, m_q, 0, \ldots, 0)$ we form the n-tuple of positive integers

$$\mu = (m_1^1, m_2^1, \ldots, m_{p_1}^1, m_1^2, \ldots, m_{p_2}^2, \ldots m_1^r, \ldots, m_{p_r}^r, m_1, m_2, \ldots, m_q) \tag{18}$$

If B_i, $i = 1, \ldots, r$ is the Borel subgroup of lower triangular matrices of $GL(p_i, \mathbb{C})$ and if B_q is the Borel subgroup of $GL(q, \mathbb{C})$, for $b \in B_q$ we first set $\tilde{b} = sbs$, where $s = s_q$ as in (14), and then set $\tilde{B}_q = \{\tilde{b} \mid b \in B_q\}$. The group $B_1 \times B_2 \times \cdots \times B_r \times \tilde{B}_q$ can then be identified with the group of all lower triangular block matrices β of the form

$$\beta = \begin{pmatrix} \boxed{b_1} & & & & \\ & \boxed{b_2} & & 0 & \\ & & \ddots & & \\ & 0 & & \boxed{b_r} & \\ & & & & \boxed{\tilde{b}} \end{pmatrix} \qquad b_i \in B_i,\ b \in B_q \tag{19}$$

where $p_1 + \cdots + p_r = p$ and $p + q = n$. It is a consequence of the Borel-Weil Theorem (see for example [10]) that for $k \geq n$ the tensor product of irreducible G_k modules

$$V^{(m^1)^k} \otimes \ldots \otimes V^{(m^r)^k} \otimes V^{(m^\vee)^k} \tag{20}$$

with the G_k-action given by (16), can be realized as the subspace of polynomial functions $f \in \mathcal{F}_k$ which, using the terminology of this paper, satisfy the covariant condition

$$f\left(\beta \begin{pmatrix} Z \\ W \end{pmatrix}\right) = \pi^{(\mu)}(\beta) f\left(\begin{pmatrix} Z \\ W \end{pmatrix}\right) \tag{21}$$

for μ as in (18) and where $\pi^{(\mu)}(\beta) = (b_1)_{11}^{m_1^1} \cdots (b)_{qq}^{m_q}$, as in (6). Since this covariant condition holds for all $k \geq n$, we realize the tensor product of irreducible G_∞-modules

$$V^{(m^1)^\infty} \otimes \ldots \otimes V^{(m^r)^\infty} \otimes V^{(m^\vee)^\infty}$$

as the inductive limit of irreducible G_k-modules (20) induced by the embeddings (17) and (3), whose elements transform according to the covariant condition (21).

For each k, let I^k denote the identity representation of G_k appearing in the tensor product

$$V^{(m^1)^k} \otimes \cdots \otimes V^{(m^r)^k} \otimes V^{(m^\vee)^k}.$$

then I^k has signature $(\underbrace{0, \ldots, 0}_{k})$ and by definition there exists a non-zero element

$$f_k \in V^{(m^1)^k} \otimes \cdots \otimes V^{(m^r)^k} \otimes V^{(m^\vee)^k}$$

such that $[R_k \otimes R_k^\vee](g) f_k = f_k$ for all $g \in G_k$. This means that f_k is *invariant* under the action $R_k \otimes R_k^\vee$ of G_k. Now it is well known from the theory of invariants (see for example [14]) that the algebra of polynomial invariants under this G_k action is generated by the pq algebraically independent polynomial functions

$$P^k_{\alpha\beta}(Z, W) = (Z\, sW^T)_{\alpha\beta} = \sum_{t=1}^{k} Z_{\alpha,t} W_{\beta,t} \quad 1 \le \alpha \le p,\ 1 \le \beta \le q. \tag{22}$$

By our realization of the $V^{(m^i)^k}$ and $V^{(m^\vee)^k}$ as G_k-modules we obviously have the isometric embedding

$$V^{(m^1)^k} \otimes \cdots \otimes V^{(m^r)^k} \otimes V^{(m^\vee)^k} \subset V^{(m^1)^{k+1}} \otimes \cdots \otimes V^{(m^r)^{k+1}} \otimes V^{(m^\vee)^{k+1}}$$

of G_k-modules into G_{k+1}-modules. It is routine to check that the appropriate diagrams commute, and as in (4) and (8) we obtain the representation $R \otimes R^\vee$ of G_∞ on $V^{(m^1)^\infty} \otimes \cdots \otimes V^{(m^r)^\infty} \otimes V^{(m^\vee)^\infty}$ as an inductive limit of representations of G_k, $k = 1, 2, 3, \ldots$. But the case of the identity representation is entirely different. For each k let $\mathcal{I}^k$ denote the one-dimensional subspace of $V^{(m^1)^k} \otimes \cdots \otimes V^{(m^r)^k} \otimes V^{(m^\vee)^k}$ spanned by the invariant vector f_k mentioned above. Then we obviously can not define the inductive limit of I^k. However we can define the *inverse* or *projective limit* of the family $\{G_k, I^k, \mathcal{I}^k\}$ as follows: For each pair of indices j, k with $j \le k$ define a continuous homomorphism $\phi^k_j : \mathcal{I}^k \longrightarrow \mathcal{I}^j$ such that

a) ϕ^j_j is the identity map for all j,

b) if $i \le j \le k$, then $\phi^k_i = \phi^k_j \circ \phi^j_i$.

Here we can take ϕ_j^k as the *truncation homomorphism*, i.e. ϕ_j^k is defined on the generators $P_{\alpha\beta}^k$ by

$$\phi_j^k(P_{\alpha\beta}^k) = P_{\alpha\beta}^j \quad \text{for } j \leq k \tag{23}$$

The *inverse limit* of the system $\{\mathcal{I}^k, \phi_j^k\}$ is then formally defined by

$$\mathcal{I}^\infty := \varprojlim \mathcal{I}^k = \left\{ (f_k) \in \prod_k \mathcal{I}^k \mid f_i = \phi_i^j(f_j) \quad \text{whenever } i \leq j \right\}$$

Concretely we can define the functions

$$P_{\alpha\beta} := P_{\alpha\beta}^\infty = \lim_{k \longrightarrow \infty} P_{\alpha\beta}^k = \sum_{t=1}^{\infty} Z_{\alpha,t} W_{\beta,t} \qquad 1 \leq \alpha \leq p,\ 1 \leq \beta \leq q \tag{24}$$

and make the following observations for each α, β:

1) $P_{\alpha,\beta}$ is well defined on

$$\mathbb{C}^{p\times\infty} \oplus \mathbb{C}^{q\times\infty^\vee} = \bigcup_{k=1}^{\infty} \left(\mathbb{C}^{p\times k} \oplus \mathbb{C}^{q\times k^\vee} \right)$$

2)$P_{\alpha,\beta}$ is <u>not</u> an element of $\mathcal{F}_\infty$, but instead lies in $\varprojlim\ \mathcal{F}_k$, the *projective limit* or *inverse limit* of the of Bargmann-Segal-Fock spaces $\mathcal{F}_k$ (for details on the projective limit representations of G_∞ see [15]).

It follows that any $f \in \mathcal{I}^\infty$ has the form

$$f = \sum C_{IJK} \prod (P_{\alpha\beta})^\gamma \tag{25}$$

where the functions $P_{\alpha\beta}$ are as defined in (24) for $1 \leq \alpha \leq p, \quad 1 \leq \beta \leq q$, the γ are non-negative integers, the sums and products in (25) are finite, and the C_{IJK} are constants with multi-indices I, J and K. Let $\pi_k : \mathcal{I}^\infty \longrightarrow \mathcal{I}^k$

denote the projection of $\mathcal{I}^\infty$ onto $\mathcal{I}^k$. Let I^∞ denote the representation of G_∞ on $\mathcal{I}^\infty$ given by the following equation

$$I^\infty(g)f = \sum C_{IJK} \prod \lim_{k\longrightarrow\infty} \left[\left(R\otimes R^{\vee}(g)P^k_{\alpha\beta} \right)^\gamma \right] \quad \text{for } g\in G_\infty \text{ and } f\in\mathcal{I}^\infty \tag{26}$$

Since $g\in G_\infty$ means that $g\in G_j$ for some j, and for $k\geq j$

$$\left[R\otimes R^{\vee}(g) \right] P^k_{\alpha\beta} = P^k_{\alpha\beta}$$

equation (26) implies that $P_{\alpha\beta}$ are G_∞- invariant, and hence $I^\infty(g)f = f$ for all $f\in\mathcal{I}^\infty$. It follows that $\pi_k\left(I^\infty(g)f\right) = \pi_k(f)$ for all $g\in G_\infty$ and $f\in\mathcal{I}^\infty$.

Recall that if $\mathcal{P}_k = \mathcal{P}(\mathbb{C}^{n\times k})$ denotes the subspace of all polynomial functions of $\mathbb{C}^{n\times k}$ then $\mathcal{P}_k$ is dense in $\mathcal{F}_k$. Let

$$\mathcal{P}_\infty = \bigcup_{k=1}^{\infty} \mathcal{P}_k$$

denote the inductive limit of $\mathcal{P}_k$, then clearly $\mathcal{P}_\infty$ is dense in $\mathcal{F}_\infty$. Let $\mathcal{P}^*_\infty$ (resp. $\mathcal{F}^*_\infty$) denote the *dual* or *adjoint* space of $\mathcal{P}_\infty$ (resp. $\mathcal{F}_\infty$). Then since $\mathcal{P}_\infty$ is dense in $\mathcal{F}_\infty$, $\mathcal{F}^*_\infty$ is dense in $\mathcal{P}^*_\infty$. By the Riesz representation theorem for Hilbert spaces, every element $f^*\in\mathcal{F}^*_\infty$ is of the form $\langle\cdot\,|f\rangle$ for some $f\in\mathcal{F}_\infty$, and the map $f^*\mapsto f$ is an anti-linear (or conjugate-linear) isomorphism. Thus we can identify $\mathcal{F}^*_\infty$ with $\mathcal{F}_\infty$ and obtain the rigged Hilbert space as the triple $\mathcal{P}_\infty\subset\mathcal{F}_\infty\subset\mathcal{P}^*_\infty$ (see [16] for the definition of rigged Hilbert spaces). Typically and element $P_{\alpha\beta}$ defined by equation (24) belongs to $\mathcal{P}^*_\infty$, and if $f\in\mathcal{F}_\infty$ then $f\in\mathcal{F}_k$ for some k, so we can define the inner product

$$\langle P_{\alpha\beta}, f\rangle = \langle \pi_k(P_{\alpha\beta}), f\rangle = \langle P^k_{\alpha\beta}, f\rangle \tag{27}$$

in fact, in the calculation of

$$P_{\alpha\beta}(D)\overline{f(\bar{Z})}\Big|_{Z=0}$$

the terms in $P_{\alpha\beta}$ whose column indices are larger than k drop off.

Theorem VI.1. *Let $V^{(m^1)^\infty}, \ldots, V^{(m^r)^\infty}$ and $V^{(m)^\infty}$ be irreducible representations of G_∞. Using the convention of Section V, let $V^{(m^\vee)^\infty}$ be the representation contragredient to $V^{(m)^\infty}$. Let I^∞ be the identity representation defined by Equation 26. Then the multiplicity of $V^{(m)^\infty}$ in the tensor product*

$$V^{(m^1)^\infty} \otimes \ldots \otimes V^{(m^r)^\infty}$$

is equal to the multiplicity of I^∞ in the tensor product

$$V^{(m^1)^\infty} \otimes \ldots \otimes V^{(m^p)^\infty} \otimes V^{(m^\vee)^\infty}.$$

Proof. From [10] we know that for sufficiently large k the multiplicity of $V^{(m)^k}$ in

$$V^{(m^1)^k} \otimes \ldots \otimes V^{(m^r)^k}$$

is equal to the multiplicity of the identity representation I^k in the augmented tensor product

$$V^{(m^1)^k} \otimes \ldots \otimes V^{(m^r)^k} \otimes V^{(m^\vee)^k}.$$

For each k let h_k denote the homomorphism sending the irreducible representation of G_k with signature $(0, \ldots, 0)$ into the G_k-module

$$V^{(m^1)^k} \otimes \ldots \otimes V^{(m^r)^k} \otimes V^{(m^\vee)^k}.$$

Then

$$\phi_j^k \circ h_k = h_j \quad \text{for } j \leq k$$

where the homomorphisms ϕ_j^k are defined as in (23). Let $(0, \ldots, 0)^\infty$ denote the signature of the representation of G_∞ as the inverse limit of irreducible representations of G_k with signature $\underbrace{(0, \ldots, 0)}_{k}$. Then we can define a homomorphism

$$h : V^{(0,\ldots,0)^\infty} \longrightarrow V^{(m^1)^\infty} \otimes \ldots \otimes V^{(m^r)^\infty} \otimes V^{(m^\vee)^\infty}$$

by

$$h(v) = \varprojlim \; h_k\left(\pi_k(v)\right) \tag{28}$$

where in Equation (28), π_k denotes the projection of $V^{(0,\ldots,0)^\infty}$ onto $V^{(0,\ldots,0)^k}$. Note that $V^{(0,\ldots,0)^k}$ or $V^{(0,\ldots,0)^\infty}$ are just the trivial G_k or G_∞ modules $\mathbb{C}$, and that the G_∞-module

$$V^{(m^1)^\infty} \otimes \ldots \otimes V^{(m^r)^\infty} \otimes V^{(m^\vee)^\infty}$$

is considered as a G_∞-submodule of the G_∞-module $\mathcal{P}_\infty^*$. As remarked in Section III, the dimension of

$$\mathrm{Hom}_{G_k}\left(V^{(0,\ldots,0)^k}, V^{(m^1)^k} \otimes \ldots \otimes V^{(m^r)^k} \otimes V^{(m^\vee)^k}\right),$$

the space of all homomorphisms intertwining $V^{(0,\ldots,0)^k}$ and $V^{(m^1)^k} \otimes \ldots \otimes V^{(m^r)^k} \otimes V^{(m^\vee)^k}$ stabilizes as k gets large. But this dimension is just the

multiplicity of I^k in $V^{(m^1)^k} \otimes \ldots \otimes V^{(m^r)^k} \otimes V^{(m^\vee)^k}$ which, in turn is equal to the multiplicity of $V^{(m)^k}$ in $V^{(m^1)^k}, \ldots, V^{(m^r)^k}$. It follows that at the (inductive) limit we have

$$\dim \left[\mathrm{Hom}_{G_\infty} \left(V^{(0,\ldots,0)^\infty}, V^{(m^1)^\infty} \otimes \ldots \otimes V^{(m^r)^\infty} \otimes V^{(m^\vee)^\infty} \right) \right]$$
$$= \dim \left[\mathrm{Hom}_{G_\infty} \left(V^{(m)^\infty}, V^{(m^1)^\infty} \otimes \ldots \otimes V^{(m^p)^\infty} \right) \right]$$

or equivalently the multiplicity of $V^{(m)^\infty}$ in the tensor product

$$V^{(m^1)^\infty} \otimes \ldots \otimes V^{(m^r)^\infty}$$

is equal to the multiplicity of I^∞ in the tensor product

$$V^{(m^1)^\infty} \otimes \ldots \otimes V^{(m^r)^\infty} \otimes V^{(m^\vee)^\infty}.$$

$\square$

Let $\{f^{m^i}_{\xi_i}\}_{\xi_i}$ be a basis of state vectors for $V^{(m^i)^\infty}$, $i = 1 \ldots r$, let $\{f^m_\xi\}_\xi$ be a basis of state vectors for $V^{(m)^\infty}$ and let $\{f^{m^*}_{\xi^*}\}_{\xi^*}$ be a basis of state vectors for $V^{(m^\vee)^\infty}$. Then

$$f^{m^1}_{\xi_1} \otimes f^{m^2}_{\xi_2} \otimes \ldots \otimes f^{m^r}_{\xi_r}$$

is a natural basis for the tensor product of irreducible representations

$$V^{(m^1)^\infty} \otimes \ldots \otimes V^{(m^r)^\infty}$$

and

$$f^{m^1}_{\xi_1} \otimes f^{m^2}_{\xi_2} \otimes \ldots \otimes f^{m^r}_{\xi_r} \otimes f^{m^*}_{\xi^*}$$

is a natural basis for the tensor product of irreducible representations

$$V^{(m^1)^\infty} \otimes \ldots \otimes V^{(m^r)^\infty} \otimes V^{(m^\vee)^\infty}$$

Let $\{\mathcal{I}_\eta\}_\eta$ be a basis for the G_∞-invariant subspace which is '*contained*' in

$$V^{(m^1)^\infty} \otimes \ldots \otimes V^{(m^r)^\infty} \otimes V^{(m^\vee)^\infty}$$

in the sense described above. If we set

$$\mathcal{I}_\eta\left(\begin{pmatrix} Z \\ W \end{pmatrix}\right) = \mathcal{I}_\eta(Z, W)$$

and consider $\mathcal{I}_\eta(Z, W)$ as a function of W, and also note that any function $f \in V^{(m^\vee)^\infty}$ is a function of W alone, then we can form the inner product, as defined in (27)

$$\langle \mathcal{I}_\eta \mid f \rangle_W = \mathcal{I}_\eta(Z, D)\overline{f(\bar{W})}|_{W=0} \tag{29}$$

and thereby obtain a function of Z.

Considering the remarks above, we adapt the statement and proof of Theorem 2.3 of [10], to our situation as follows

Theorem VI.2. *Let*

$$\tilde{f}_\xi^{m,\eta}(Z) = \langle \mathcal{I}_\eta(Z, W) \mid f_{\xi^*}^{m^*}(W)\rangle_W = \mathcal{I}_\eta(Z, D)\overline{f_{\xi^*}^{m^*}(\bar{W})}|_{W=0}$$

Then $\{\tilde{f}_\xi^{m,\eta}\}_\xi$ *is an isomorphic image of* $\{f_\xi^m\}_\xi$ *in* $V^{(m^1)^\infty} \otimes \ldots \otimes V^{(m^r)^\infty}$ *indexed by the multiplicity label* η *and we have the following relation of*

Clebsch-Gordan coefficients

$$\langle \tilde{f}_{\xi}^{m,\eta} | f_{\xi_1}^{m^1} f_{\xi_2}^{m^2} \ldots f_{\xi_r}^{m^r} \rangle = \langle \mathcal{I}_{\eta} | f_{\xi_1}^{m^1} f_{\xi_2}^{m^2} \ldots f_{\xi_r}^{m^r} f_{\xi^*}^{m^*} \rangle \tag{30}$$

Proof. To first show that $\tilde{f}_{\xi}^{m,\eta}(Z)$ in fact lies in $V^{(m^1)^\infty} \otimes \ldots \otimes V^{(m^r)^\infty}$ it is sufficient to show (by the Borel-Weil theorem) that if $b = (b_1, \ldots, b_r) \in B_1 \times \cdots \times B_r$ then, as in Equation (21)

$$\tilde{f}_{\xi}^{m,\eta}(bZ) = \pi^{\mu(m^1)}(b_1) \cdots \pi^{\mu(m^r)}(b_r) \tilde{f}_{\xi}^{m,\eta}(Z)$$

But since $\mathcal{I}_{\eta}$ '*lies* ' in

$$V^{(m^1)^\infty} \otimes \ldots \otimes V^{(m^r)^\infty} \otimes V^{(m^\vee)^\infty}$$

it transforms covariantly with respect to the Borel subgroup defined in Equation (19) so we have

$$\begin{aligned}
\tilde{f}_{\xi}^{m,\eta}(bZ) &= \langle \mathcal{I}_{\eta}\ (bZ, W) \mid f_{\xi^*}^{m^*}(W) \rangle_W \\
&= \langle \mathcal{I}_{\eta}\ (bZ, Id\ W) \mid f_{\xi^*}^{m^*}(W) \rangle_W \quad \text{where } Id \text{ is the } q \times q \text{ identity matrix} \\
&= \langle\ \pi^{\mu}(\beta)\ \mathcal{I}_{\eta}(Z, W) \mid f_{\xi^*}^{m^*}(W) \rangle_W \quad \text{where } \beta = b \times Id \\
&= \pi^{\mu}(\beta)\ \langle\ \mathcal{I}_{\eta}(Z, W) \mid f_{\xi^*}^{m^*}(W) \rangle_W \quad \text{the inner product is linear in the first argument} \\
&= \pi^{\mu(m^1)}(b_1) \cdots \pi^{\mu(m^r)}(b_r) \tilde{f}_{\xi}^{m,\eta}(Z) \quad \text{by Equation (21), as desired.}
\end{aligned}$$

We next show that the $\{\tilde{f}_{\xi}^{m,\eta}\}_{\xi}$ transform under the representation $R^{(m)}$ in the same manner as the $\{f_{\xi}^{m}\}_{\xi}$. Since $\mathcal{I}_{\eta}(Z, W)$ is invariant with respect to the action $R \otimes R^{\vee}$ of G_∞ we have $\mathcal{I}_{\eta}(Zg, W) = \mathcal{I}_{\eta}(Z, W s {g^{-1}}^{\vee} s)$ which can succinctly be written as $R(g)\mathcal{I}_{\eta}(Z, W) = R^{\vee}(g^{-1})\mathcal{I}_{\eta}(Z, W)$. We also have that

$$R^{(m)}(g) f_{\xi}^{m} = \sum_{\xi'} \mathcal{D}_{\xi\xi'}^{m}(g) f_{\xi'}^{m}$$

where the $\mathcal{D}^m_{\xi\xi'}$ are the D-functions for the representation $R^{(m)}$. Now for any $g \in G_\infty$ we can assume that $g \in U(k)$ for some k, so that $g^\vee = \bar{g}$. Hence $\mathcal{D}^m_{\xi\xi'}(g^\vee) = \overline{\mathcal{D}^m_{\xi\xi'}(g)}$, and it follows from the definitions of the symbols involved that

$$R^{(m)\vee}(g) f^{m^*}_{\xi^*} = \sum_{\xi^{*\prime}} \overline{\mathcal{D}^m_{\xi\xi'}(g)} f^{m^*}_{\xi^{*\prime}}$$

Thus we seek to show that

$$R^{(m)}(g) \tilde{f}^{m,\eta}_{\xi} = \sum_{\xi'} \mathcal{D}^m_{\xi\xi'}(g) \tilde{f}^{m,\eta}_{\xi'}$$

By the preceding remarks and the definition of $\tilde{f}^{m,\eta}_{\xi}$ we then have

$$\begin{aligned}
R^{(m)}(g)\tilde{f}^{m,\eta}_{\xi}(Z) &= \langle \mathcal{I}_\eta\ (Zg, W) \mid f^{m^*}_{\xi^*}(W)\rangle_W \\
&= \langle R^{(m)}(g)\, \mathcal{I}_\eta\ (Z, W) \mid f^{m^*}_{\xi^*}(W)\rangle_W \\
&= \langle R^{(m)\vee}(g^{-1})\, \mathcal{I}_\eta\ (Z, W) \mid f^{m^*}_{\xi^*}(W)\rangle_W \\
&= \langle \mathcal{I}_\eta\ (Z, W) \mid R^{(m)\vee}(g) f^{m^*}_{\xi^*}(W)\rangle_W \qquad \text{since the representation is unitary} \\
&= \langle \mathcal{I}_\eta\ (Z, W) \mid \sum_{\xi^{*\prime}} \overline{\mathcal{D}^m_{\xi\xi'}(g)} f^{m^*}_{\xi^{*\prime}}(W)\rangle_W \\
&= \sum_{\xi'} \mathcal{D}^m_{\xi\xi'}(g)\, \langle \mathcal{I}_\eta\ (Z, W) \mid f^{m^*}_{\xi^{*\prime}}(W)\rangle_W \qquad \text{by conjugate linearity} \\
&= \sum_{\xi'} \mathcal{D}^m_{\xi\xi'}(g) \tilde{f}^{m,\eta}_{\xi'}
\end{aligned}$$

Finally we have

$$\begin{aligned}
\langle \mathcal{I}_\eta | f^{m^1}_{\xi_1} f^{m^2}_{\xi_2} \ldots f^{m^r}_{\xi_r} f^{m^*}_{\xi^*} \rangle &= \langle \mathcal{I}_\eta | f^{m^*}_{\xi^*} f^{m^1}_{\xi_1} f^{m^2}_{\xi_2} \ldots f^{m^r}_{\xi_r} \rangle \\
&= \mathcal{I}_\eta(D, D) \overline{f^{m^*}_{\xi^*}(\bar{W})}\, \overline{f^{m^1}_{\xi_1}(\bar{Z})} \ldots \overline{f^{m^r}_{\xi_r}(\bar{Z})}|_{(Z,W)=(0,0)} \\
&= \left[\mathcal{I}_\eta(D, D) \overline{f^{m^*}_{\xi^*}(\bar{W})}\right]\, \overline{f^{m^1}_{\xi_1}(\bar{Z})} \ldots \overline{f^{m^r}_{\xi_r}(\bar{Z})}|_{(Z,W)=(0,0)} \\
&= \tilde{f}^{m,\eta}_{\xi}(D) \overline{f^{m^1}_{\xi_1}(\bar{Z})} \ldots \overline{f^{m^r}_{\xi_r}(\bar{Z})}|_{Z=0} \\
&= \langle \tilde{f}^{m,\eta}_{\xi} | f^{m^1}_{\xi_1} f^{m^2}_{\xi_2} \ldots f^{m^r}_{\xi_r} \rangle
\end{aligned}$$

which is Equation (30). $\square$

VII Example

We illustrate the techniques described in this paper with the example

$$(7, 1, \overrightarrow{0}) \subset (1, \overrightarrow{0}) \otimes (2, \overrightarrow{0}) \otimes (2, \overrightarrow{0}) \otimes (3, \overrightarrow{0})$$

considered in (11) of Section III. By the results of Theorem VI.2 and Equation (21) we seek algebraically independent polynomials of the form

$$P\left(\begin{pmatrix} Z \\ W \end{pmatrix}\right) = \sum C_{IJK} \prod (P_{\alpha\beta})^{\gamma} \quad \alpha = 1, 2, 3, 4 \quad \beta = 1, 2 \tag{31}$$

that satisfy the covariant condition

$$P\left(\beta \begin{pmatrix} Z \\ W \end{pmatrix}\right) = \pi^{(\mu)}(\beta) f\left(\begin{pmatrix} Z \\ W \end{pmatrix}\right) \tag{32}$$

where $\mu = (1, 2, 2, 3, 7, 1)$ and

$$\beta = \begin{pmatrix} b_1 & & & & & \\ & b_2 & & & 0 & \\ & & b_3 & & & \\ & & & b_4 & & \\ & 0 & & & b_5 & b* \\ & & & & & b_6 \end{pmatrix} \quad b_i,\, b* \in C.$$

If $\mathbf{D}$ is the diagonal subgroup and $\mathbf{Z}^+$ is the upper triangular unipotent subgroup, then $\beta \in \mathbf{D}\mathbf{Z}^+$ so we can first reduce the problem by solving (32) for the diagonal subgroup $\mathbf{D}$ which consists of elements of the form

$$d = \begin{pmatrix} b_1 & & & & & \\ & b_2 & & & 0 & \\ & & b_3 & & & \\ & & & b_4 & & \\ & 0 & & & b_5 & \\ & & & & & b_6 \end{pmatrix} \quad b_i \in \mathbb{C}.$$

Hence we seek polynomials of the form

$$P = P_{11}^{\ell_{11}} P_{12}^{\ell_{12}} P_{21}^{\ell_{21}} \cdots P_{41}^{\ell_{41}} P_{42}^{\ell_{42}}$$

that satisfy

$$P\left(d\begin{pmatrix} Z \\ W \end{pmatrix}\right) = \pi^{(\mu)}(d) f\left(\begin{pmatrix} Z \\ W \end{pmatrix}\right), \forall\ d \in \mathbf{D}.$$

This leads us to the system

$$\begin{cases} \ell_{11} + \ell_{12} = 1 \\ \ell_{21} + \ell_{22} = 2 \\ \ell_{31} + \ell_{32} = 2 \\ \ell_{41} + \ell_{42} = 3 \\ \ell_{11} + \ell_{21} + \ell_{31} + \ell_{41} = 7 \\ \ell_{12} + \ell_{22} + \ell_{32} + \ell_{42} = 1 \end{cases}$$

which gives us the following set of polynomials that transform covariantly with respect to the diagonal subgroup $\mathbf{D}$;

$$\begin{aligned} P_1 =& P_{11}P_{21}P_{22}P_{31}^2P_{41}^3 \\ P_2 =& P_{11}P_{21}^2P_{31}P_{32}P_{41}^3 \\ P_3 =& P_{11}P_{21}^2P_{31}^2P_{41}^2P_{42} \\ P_4 =& P_{12}P_{21}^2P_{31}^2P_{41}^3. \end{aligned}$$

Next, from (31) and (32) we seek functions of the form

$$P = C_1P_1 + C_2P_2 + C_3P_3 + C_4P_4 \tag{33}$$

that transform covariantly with respect to the upper triangular unipotent

subgroup $\mathbf{Z}^+$ which consists of elements of the form

$$Z^+ = \begin{pmatrix} 1 & & & & & \\ & 1 & & & 0 & \\ & & 1 & & & \\ & & & 1 & & \\ & 0 & & & 1 & b* \\ & & & & & 1 \end{pmatrix} \quad b* \in \mathbb{C}.$$

Checking this condition on $P_1,\ P_2,\ P_3$ and P_4 we see that

$$P_1 \left(Z^+ \begin{pmatrix} Z \\ W \end{pmatrix} \right) = P_1 + b * P_{11} P_{21}^2 P_{31}^2 P_{41}^3$$

$$P_2 \left(Z^+ \begin{pmatrix} Z \\ W \end{pmatrix} \right) = P_2 + b * P_{11} P_{21}^2 P_{31}^2 P_{41}^3$$

$$P_3 \left(Z^+ \begin{pmatrix} Z \\ W \end{pmatrix} \right) = P_3 + b * P_{11} P_{21}^2 P_{31}^2 P_{41}^3$$

$$P_4 \left(Z^+ \begin{pmatrix} Z \\ W \end{pmatrix} \right) = P_4 + b * P_{11} P_{21}^2 P_{31}^2 P_{41}^3$$

In order that

$$P \left(Z^+ \begin{pmatrix} Z \\ W \end{pmatrix} \right) = P \left(\begin{pmatrix} Z \\ W \end{pmatrix} \right) \quad \forall Z^+ \in \mathbf{Z}^+$$

we must have $C_1 + C_2 + C_3 + C_4 = 0$. Thus a convenient basis of G_∞-invariants in this tensor product can be chosen as

$$\begin{aligned} \mathcal{I}_1 = P_1 - P_2 =& P_{11} P_{21} P_{22} P_{31}^2 P_{41}^3 - P_{11} P_{21}^2 P_{31} P_{32} P_{41}^3 \\ \mathcal{I}_2 = P_2 - P_3 =& P_{11} P_{21}^2 P_{31} P_{32} P_{41}^3 - P_{11} P_{21}^2 P_{31}^2 P_{41}^2 P_{42} \\ \mathcal{I}_3 = P_3 - P_4 =& \quad P_{11} P_{21}^2 P_{31}^2 P_{41}^2 P_{42} - P_{12} P_{21}^2 P_{31}^2 P_{41}^3. \end{aligned}$$

Note that the space of invariants has dimension three, which is the multiplicity of $(7, 1, \overrightarrow{0})$ computed earlier.

Now a natural basis for the G_∞-invariant subspace with signature $(1, \overrightarrow{0})$ contained in $\mathcal{F}_\infty$ as described in Section VI is given by $\{Z_{1i}\}_{i=1}^\infty$. Similarly $\{Z_{2i}Z_{2j}\}_{i,j=1}^\infty$, $\{Z_{3i}Z_{3j}\}_{i,j=1}^\infty$ and $\{Z_{4i}Z_{4j}Z_{4k}\}_{i,j,k=1}^\infty$ are natural basis for the subspaces $(2, \overrightarrow{0})$, $(2, \overrightarrow{0})$ and $(3, \overrightarrow{0})$, respectively, and an element of $(7, 1, \overrightarrow{0})^\vee$ is its lowest weight vector $w_{11}^6 \det \left(\begin{smallmatrix} w_{22} & w_{21} \\ w_{12} & w_{11} \end{smallmatrix}\right)$. Thus an example of a basis element for the tensor product

$$(1, \overrightarrow{0}) \otimes (2, \overrightarrow{0}) \otimes (2, \overrightarrow{0}) \otimes (3, \overrightarrow{0}) \otimes (7, 1, \overrightarrow{0})^\vee$$

would be

$$Z_{11}Z_{21}^2Z_{31}^2Z_{41}^3(W_{11}^7W_{22} - W_{11}^6W_{21}) \tag{34}$$

and to compute a Clebsch-Gordan coefficient we compute the inner product of (34) with, for example $\mathcal{I}_1$.

$$\begin{aligned} &\langle \mathcal{I}_1 \,|\, Z_{11}Z_{21}^2Z_{31}^2Z_{41}^3(W_{11}^7W_{22} - W_{11}^6W_{21})\rangle = \\ &\qquad \left[P_{11}P_{21}P_{22}P_{31}^2P_{41}^3 - P_{11}P_{21}^2P_{31}P_{32}P_{41}^3\right](D) \\ &\qquad\qquad Z_{11}Z_{21}^2Z_{31}^2Z_{41}^3(W_{11}^7W_{22} - W_{11}^6W_{21})\big|_{(Z,W)=(0,0)} \end{aligned}$$

We remark that in the above computation, for example the product

$$P_{11}P_{21}P_{22}P_{31}^2P_{41}^3(D) = \left(\sum_{t=1}^\infty Z_{1t}W_{1t}\right)\cdots\left(\sum_{t=1}^\infty Z_{4t}W_{1t}\right)^3(D)$$

need only be evaluated up to $t = 2$ since those terms whose column indices are larger than two evaluate to zero in the above inner product. This is routinely accomplished using a computer algebra system, such as *Maple*.

VIII Conclusion

We have shown how the multiplicity problem the Clebsch-Gordan coefficients in the decomposition of r-fold tensor products of irreducible tame representations of $U(\infty)$ can be restated in terms of $U(\infty)$-invariants. Thus all the theorems for $U(k)$ treated in [10] can be generalized to $U(\infty)$. Actually the computational aspect of the problems are much simpler with this new approach and one can use computers to obtain invariant polynomials, and by differentiating these polynomials compute Clebsch-Gordan and Racah coefficients.

Acknowledgments

R. M. Howe was partially supported by the University of Wisconsin-Eau Claire Office of University Research.

T. Ton-That was partially supported while on Developmental Research Assignment at the University of Iowa.

References

[1] I. Segal. The structure of a class of representations of the unitary group on a Hilbert space. *Proceedings of the American Mathematical Society*, 8:197–203, 1957.

[2] A. Kirillov. Representations of an infinite dimensional unitary group. *Dodkladi Akad. Nauk SSSR*, 14(212):1355–1358, 1973.

[3] S. Stratila and D. Voiculescu. *Representations of AF Algebras and the Group $U(\infty)$*. Number 486 in Lecture Notes in Mathematics. Springer-Verlag, New York.

[4] D. Pickrell. Decompositions of regular representations of $U(H)_\infty$. *Pacific Journal of Mathematics*, 128:319–332, 1987.

[5] G. I. Ol'shanski. Representations of infinite-dimensional classical groups, limits of enveloping algebras, and Yangians.

[6] G. I. Ol'shanski. The method of holomorphic extensions in the theory on unitary representations of infinite dimensional classical groups. *Functional Analysis Appl.*, 22(4), 1988.

[7] G. I. Ol'shanski. Unitary representations of infinite-dimensional pairs (G, K) and the formalism of R. Howe. In A. M. Vershik and D. P. Zelobenko, editors, *Representations of Lie Groups and Related Topics*. Gordon and Breach Scientific Publishers, New York, 1991.

[8] I. M. Gelfand and M. I. Graev. Principal representations of the group $U(\infty)$. In A. M. Vershik and D. P. Zelobenko, editors, *Representations of Lie Groups and Related Topics*. Gordon and Breach Scientific Publishers, New York, 1991.

[9] V. Kac. Highest weight representations of infinite dimensional Lie algebras. In *Proceedings of ICM, Helsinki*, 1355–1358, 1978.

[10] W. H. Klink and T. Ton-That. Multiplicity, invariants, and tensor products of compact groups. *Journal of Mathematical Physics*, 37:6468–6485, Dec. 1996.

[11] T. Ton-That. Reciprocity theorems for holomorphic representations of some infinite dimensional groups. To appear in Helvetica Physica Acta, 1999.

[12] T. Ton-That. Invariant theory for tame representations of infinite dimensional classical groups. AMS Abstract of the Special Session on Invariant Theory, AMS Meeting number 924, Montreal Canada, September 1997.

[13] D. P. Zhelobenko. *Compact Lie Groups and their Representations*, volume 40 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, Rhode Island, 1973.

[14] H. Weyl. *The Classical Groups, Their Invariants and Representations*. Princeton University Press, Princeton, New Jersey, second edition, 1946.

[15] R. M. Howe. Dual representations of $gl(\infty)$ and decomposition of Fock spaces. *Journal of Physics (A)*, 2757–2781, 1997.

[16] I. M. Gelfand and G. E. Shilov. *Generalized Functions*,Vol. 4, Academic Press, New York 1988 .